Morningstar Risk Decomposition
Methodology

Morningstar Quantitative Research
April 2021
Version 2.0

Contents

Authors

Gordon Sutton, PhD
Quantitative Analyst
+61 2 9274 0324
Gordon.Sutton@morningstar.com

Dave Sahota, PhD, CFA
Lead Quantitative Researcher
+1 312-244-7163
Dave.Sahota@morningstar.com

Patrick Caldon, PhD
Director of Quantitative Research
+61 2 9276 4473
Patrick.Caldon@morningstar.com

Overview

Risk is inherent to investing. Investment professionals rely on the decomposition of portfolio risk into factors to guide their investment decisions. Morningstar's Risk Decomposition tool decomposes portfolio risk into their individual factor components, allowing users to view the sources of total and active risk within each portfolio. With this tool, users can dissect portfolio risk along the dimensions of the factors in Morningstar's Global Risk Model and develop an understanding of each factor's marginal and total risk contribution.

Key Takeaways

- User can dissect portfolio risk (measured by standard deviation) along the dimensions of the factors in Morningstar's Global Risk Model.
- User can decompose each portfolio's total risk as well as its active risk (risk relative to a benchmark).
- User can view each factor's marginal and total risk contribution, in both absolute and percentage terms.

Risk Model Definitions

Morningstar defines risk as the standard deviation of portfolio returns. Risk Decomposition dissects portfolio risk along the dimensions of the factors in Morningstar's Global Risk Model, a proprietary fundamental-based linear factor model. All factor models rely on a simple linear representation of asset returns. In the following section, we will outline the basic structure of linear factor models.

Asset Return

We model asset returns at every period t as follows:

$$R = X \cdot P_F + S \qquad (1)$$

under the constraint that

$$C \cdot P_F = 0 \qquad (2)$$

where:

- R = N x 1 vector of asset returns
- X = N x K matrix of asset-level factor exposures
- P_F = K x 1 vector of factor returns (factor premia)
- S = N x 1 vector of asset-level specific returns (residual returns)
- C = 2 x K vector of factor premia constraint weights

In the first row of C, the i-th element, C_{1i}, is a sector constraint weight for factor i when factor i is in the sector factors and otherwise is 0. In the second row, C_{2j} is similarly a region weight for factor j when factor j is in the region factors and otherwise is 0.

The first constraint in C ensures the weighted sum of sector premia equals zero, where the weights are proportional to the market exposures of the sectors, i.e., the sum of asset exposures to the sectors, weighted by each asset's market capitalization. The second constraint in C ensures the weighted sum of region premia equals zero, where the weights are proportional to the market exposures of the regions.

Equations (1) and (2) are solved as a constrained weighted regression with the objective of minimizing S^TWS, where the weights matrix W is an N x N diagonal matrix with the i-th diagonal entry proportional to the square-root of asset i's market capitalization. The square-root of capitalization is used because it has empirically been found to be approximately proportional to the inverse of the variance of specific returns.

We make the following two assumptions:

- The specific returns S_i are uncorrelated with the factor returns P_F,
  - Cov (S_i, P_{Fj}) = 0, for all i and j.
- Asset i's specific return S_i are uncorrelated with asset j's specific return S_j,
  - Cov (S_i , S_j) = 0, for all $i \neq j$

The expanded matrix form of Equations (1) and (2) are

$$\begin{bmatrix} R_1 \\ \vdots \\ R_N \end{bmatrix} = \begin{bmatrix} X_{11} & \dots & X_{1K} \\ \vdots & \ddots & \vdots \\ X_{N1} & \dots & X_{NK} \end{bmatrix} \cdot \begin{bmatrix} P_{F1} \\ \vdots \\ P_{FK} \end{bmatrix} + \begin{bmatrix} S_1 \\ \vdots \\ S_N \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & C_{sector} & 0 & 0 \\ 0 & 0 & C_{region} & 0 \end{bmatrix} \cdot \begin{bmatrix} P_{F1} \\ \vdots \\ P_{FK} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In the case of the Morningstar Global Equity Risk Model, the estimation universe is approximately 7,000 stocks and there are 37 risk factors. In the above, N ≈ 7,000 and K = 37. These numbers and the types of factors will change depending on the model.

Asset Risk

We can complete the definition of the risk model by expressing the N x N variance-covariance matrix of asset returns V as:

$$V = X \cdot F \cdot X^T + \Delta \qquad (3)$$

where:

- V = N x N variance-covariance matrix of asset returns
- X = N x K matrix of asset-level factor exposures
- F = K x K variance-covariance matrix of the factor returns (factor premia)
- Δ = N x N variance matrix of the specific returns S (diagonal matrix of specific variance)

The expanded matrix form of Equation (3) is

$$\begin{bmatrix} \sigma_1^2 & \dots & \sigma_{1N} \\ \vdots & \ddots & \vdots \\ \sigma_{N1} & \dots & \sigma_N^2 \end{bmatrix} = X \cdot \begin{bmatrix} \sigma_{(F)1}^2 & \dots & \sigma_{(F)1K} \\ \vdots & \ddots & \vdots \\ \sigma_{(F)K1} & \dots & \sigma_{(F)K}^2 \end{bmatrix} \cdot X^T + \begin{bmatrix} \sigma_{(S)1}^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_{(S)N}^2 \end{bmatrix}$$

We leave discussion on the estimation of V, F and Δ till later.

Factor Mimicking Portfolios

Factor mimicking portfolios are portfolios that have returns that mimic the risk factor premia we have defined through the risk model. They typically contain both long and short positions.

The risk model estimates the constrained linear multiple regression shown in Equations (1) and (2). Given exposures to a set of factors, we estimate daily factor returns in a cross-sectional multivariate regression of asset returns on the factor exposures, using the stocks in our estimation universe. The estimated factor returns minimize the sum of squared specific returns while adhering to the constraint. This constrained, weighted least squares regression, leads to the following matrix expression for the estimated factor premia:

$$P_F = (X^TWX)^{-1}X^TW \cdot R - (X^TWX)^{-1}C^T(C(X^TWX)^{-1}C)^{-1}C(X^TWX)^{-1}X^TW \cdot R \qquad (4)$$

where:

- P_F = K x 1 vector of factor returns (factor premia)
- X = N x K matrix of asset-level factor exposures
- W = N x N diagonal asset-weighting matrix of weighted least squares
- R = N x 1 vector of asset returns
- C = 2 x K vector of factor premia constraint weights

The first term is the unconstrained ordinary least squares solution and the second term is the result of incorporating the constraint given in Equation (2).

If we define

$$M = (X^TWX)^{-1}X^TW - (X^TWX)^{-1}C^T(C(X^TWX)^{-1}C^T)^{-1}C(X^TWX)^{-1}X^TW \qquad (5)$$

then Equation (3) becomes

$$P_F = M \cdot R \qquad (6)$$

where:

- M = K x N matrix that constructs factor returns from asset returns, and as above:
- P_F = K x 1 vector of factor returns (factor premia)
- R = N x 1 vector of asset returns

The factor return for factor j , P_{Fj}, is simply a weighted sum of asset returns:

$$P_{Fj} = M_{j,1} * R_1 + M_{j,2} * R_2 + \cdots + M_{j,N} * R_N = \sum_{i=1}^{N} M_{j,i} * R_i \qquad (7)$$

In this form, we can interpret P_{Fj} as the return to a portfolio, with portfolio weights $M_j = (M_{j,1}, M_{j,2}, \dots, M_{j,N})$. Or conversely, we can interpret M_j as the factor mimicking portfolio for factor j.

We want each factor mimicking portfolio to mimic the returns of the factor being mimicked; to be orthogonal to all other factors, meaning that each factor mimicking portfolio has unit exposure to the factor it mimics and zero exposure to all other factors; and to have minimal residual risk. From Equation (6) and (7), M_j reproduces the factor premium for factor j, P_{Fj}. The constrained regression gives the set of P_{Fj} that minimizes the weighted sum of squared specific returns, in particular, S^TWS. If we incrementally change M, i.e., the weights of any of the factor mimicking portfolios, then we change some P_{Fj} and S will no longer be minimized. Hence, M_j produce minimal residual risk. We next delve into the constrained regression to argue that the factor mimicking portfolios effectively have unit net exposure to the factor they mimic and zero net exposure to every other factor.

We first note that the effects of the sector and region constraints are to force the market-weighted sector and region mean returns into the equity market premium, leaving the sector and region premia to model the deviations of each specific sector or region from the sector or region market-weighted mean return.

The K portfolio exposures for the K factor mimicking portfolios are given by $M \cdot X$, which equals the identity matrix minus a complicated term. The complicated term subtracts an exposure that is proportional to each sector's market-capitalization from the sector factor mimicking portfolios, and an exposure that is proportional to each region's market-capitalization from the region factor mimicking portfolios. Due to the constraints in Equation (2), the returns from these exposures to the sector/region, in proportion to their market- capitalization weights, are zero. As such, although the exposures for a sector or region factor mimicking portfolio j, do not equal one for factor j and zero otherwise, the effective net exposures do, after removing these contributions that are constrained to produce zero net return. So, the factor mimicking portfolios are effectively orthogonal.

The constraints hold at each time point the regression is calculated and the constraint weights evolve only slowly, so this effective exposure orthogonality holds ex-post (backwards looking) and holds approximately ex-ante (forwards looking).

Portfolio Return

A portfolio is described by a N x 1 vector w_P that gives the portfolio's holding-weights in N assets. The portfolio's K x 1 factor exposures x_P is given by the product of the asset-level factor exposures X^T and the holding weights w_P:

$$x_P = X^T \cdot w_P \quad (8)$$

The portfolio return r_P is given by:

$$r_P = x_P^T \cdot P_F + w_P^T \cdot S \quad (9)$$

The active return of a portfolio r_{PA} is the difference between the portfolio return and the benchmark return, for a benchmark with weights w_B. The active return is given by:

$$r_{PA} = x_{PA}^T \cdot P_F + w_{PA}^T \cdot S = (x_P - x_B)^T \cdot P_F + (w_P - w_B)^T \cdot S \quad (10)$$

where the active asset weights w_{PA} is a N x 1 vector of the portfolio holding weights minus benchmark holding weights defined as:

$$w_{PA} = w_P - w_B \quad (11)$$

and x_{PA} is the K x 1 active portfolio factor exposures given by the product of the asset-level factor exposures X^T and the active asset weights w_{PA}:

$$x_{PA} = X^T \cdot w_{PA} \quad (12)$$

Portfolio Risk

The portfolio variance, i.e., the squared portfolio risk, is given by:

$$\sigma_P^2 = x_P^T \cdot F \cdot x_P + w_P^T \cdot \Delta \cdot w_P = w_P^T \cdot V \cdot w_P \quad (13)$$

where the total risk has been separated into common-factor and asset-specific components. This works because factor risks and specific risks are uncorrelated, which results from the constrained regression.

A similar formula is derived for active risk, commonly known as tracking error σ_{PA}, which measures the relative risk of the portfolio to a selected benchmark:

$$\sigma_{PA}^2 = x_{PA}^T \cdot F \cdot x_{PA} + w_{PA}^T \cdot \Delta \cdot w_{PA} = w_{PA}^T \cdot V \cdot w_{PA} \quad (14)$$

Portfolio Risk Decomposition

There are multiple common measures of portfolio risk, including standard deviation, variance, Value at Risk, and beta. Morningstar's Risk Decomposition relies on a well-established concept called Marginal Contribution to Risk, which defines risk as the standard deviation of the portfolio return. We define the concept of Marginal Contribution to Risk in the following sections. Our presentation closely follows that in Chapter 3 of Grinold and Kahn [1].

Asset-Level Marginal Risk Contribution

The marginal impact on risk at the asset level is measured by the partial derivative of the risk with respect to the asset holding. This type of sensitivity analysis allows us to see which assets have the largest impact on portfolio risk, as defined by the standard deviation of portfolio returns.

We can compute marginal contributions to total risk and active risk from Equations (13) and (14). The N x 1 vector of marginal contributions to total risk at the asset level ($\mathrm{MCTR_A}$) is:

$$\mathrm{MCTR_A} = \frac{\partial \sigma_P}{\partial w_P^T} = \frac{V \cdot w_P}{\sigma_P} \qquad (15)$$

Equation (14) is derived by applying the chain rule when differentiating $\sigma_P = (w_P^T \cdot V \cdot w_P)^{1/2}$.

Using the notation $x(i)$ to refer to the i-th component of a vector x, $\mathrm{MCTR_A}(i)$ is the partial derivative of σ_P with respect to $w_P(i)$. $\mathrm{MCTR_A}(i)$ can be thought of as the approximate change in portfolio risk percentage given a 1 percent increase in the holding of asset i, financed by decreasing cash holding by 1 percent.

To first order,

$$\Delta\sigma_P \approx \Delta w_P^T \cdot \mathrm{MCTR_A} \qquad (16)$$

In a similar way, we can define the marginal contribution to active risk ($\mathrm{MCAR_A}$) as:

$$\mathrm{MCAR_A} = \frac{\partial \sigma_{PA}}{\partial w_{PA}^T} = \frac{V \cdot w_{PA}}{\sigma_{PA}} \qquad (17)$$

The active risk case includes the corner case of the portfolio completely comprising its benchmark, such that w_{PA} and σ_{PA} both equal zero, which warrants some consideration. As a decomposition tool, the $\mathrm{MCAR_A}$ for an active portfolio with zero risk is zero. As a construction tool, we can take the limit of incrementing the active portfolio from $w_{PA} = 0$ in a given direction, αw_0, as α approaches 0. This leads to

$$\lim_{\alpha \to 0} \left. \frac{\partial \sigma_{PA}}{\partial w_{PA}^T} \right|_{w_{PA} = \alpha W_0} = \frac{V \cdot w_0}{\sqrt{w_o^T \cdot V \cdot w_0}} \qquad (18)$$

which needs to be evaluated for the whole direction of active portfolio change, w_0, rather than separately for each changed asset, then summing their contributions; otherwise covariances are missed.

Factor-Level Marginal Risk Contribution

The marginal impact on risk at the factor level is obtained similarly, this time taking the partial derivative of the risk with respect to factor exposure, rather than to asset holdings. Using Equations (13) and (14) with the factor component separated from the asset-holdings component, $\mathrm{MCTR_F}$ and $\mathrm{MCAR_F}$ are obtained as

$$\mathrm{MCTR_F} = \frac{\partial \sigma_P}{\partial x_P^T} = \frac{F \cdot x_P}{\sigma_P} \qquad (19)$$

and

$$\mathrm{MCAR_F} = \frac{\partial \sigma_{PA}}{\partial x_{PA}^T} = \frac{F \cdot x_{PA}}{\sigma_{PA}} \qquad (20)$$

The same discussion as for $\mathrm{MCAR_A}$ about the corner case of the active portfolio completely following its benchmark applies to $\mathrm{MCAR_F}$.

Note that the $\mathrm{MCTR_F}$ and $\mathrm{MCAR_F}$ give the marginal contributions to risk due to small changes in the exposure to each risk factor, while keeping all else, i.e., the other factors and all the asset holdings constant. Thus, they are useful for description, rather than as measures to be directly enacted upon.

Asset-Level Risk Decomposition

There are different ways to decompose risk. We can use the marginal contributions at the asset level to define a decomposition of total risk. First note the mathematical relationship:

$$w_P^T \cdot \mathrm{MCTR_A} = w_P^T \cdot \left(\frac{V \cdot w_P}{\sigma_P}\right) = \sigma_P \qquad (21)$$

Equation (19) implies we can decompose the total risk, σ_P, linearly using the $\mathrm{MCTR_A}$. More rigorously, we can write w_P as $\widetilde{w}_P \alpha_P$, where $\widetilde{w}_p^T \cdot \widetilde{w}_P = 1$; let $\widetilde{\sigma}_P$ be the risk for the portfolio with holdings $\widetilde{w}_P$, such that $\sigma_P = \widetilde{\sigma}_P \alpha_P$; and then rewrite $\mathrm{MCTR_A}$ as

$$\mathrm{MCTR_A} = \frac{V \cdot \widetilde{w}_P \alpha_P}{\sqrt{\alpha_P \widetilde{w}_P^T \cdot V \cdot \widetilde{w}_P \alpha_P}} = \frac{V \cdot \widetilde{w}_P}{\sqrt{\widetilde{w}_P^T \cdot V \cdot \widetilde{w}_P}} = \frac{V \cdot \widetilde{w}_P}{\widetilde{\sigma}_P} \qquad (22)$$

and note that the $\mathrm{MCTR_A}$ depends only on the direction of the portfolio, not the magnitude. If we then integrate $d(\alpha \widetilde{w}_P^T) \cdot \mathrm{MCTR_A}$ from $\alpha = 0$ to α_P, we obtain the Contribution to Total Risk ($\mathrm{CTR_A}$) as

$$\mathrm{CTR_A} = \int_{\mathbf{0}}^{\mathbf{w_P}} \mathrm{MCTR_A^T} \cdot dw = \int_{\mathbf{0}}^{\alpha_P} \mathrm{MCTR_A^T} \cdot \widetilde{w}_P d\alpha = \widetilde{w}_P^T \cdot \mathrm{MCTR_A} \int_{\mathbf{0}}^{\alpha_P} d\alpha = \alpha_P \widetilde{w}_P^T \cdot \mathrm{MCTR_A} = w_P^T \cdot \mathrm{MCTR_A} \qquad (23)$$

It follows that the amount of total risk we can attribute to asset i, $\mathrm{CTR_A}(i)$, is

$$\mathrm{CTR_A}(i) = w_P^T(i) \cdot \mathrm{MCTR_A}(i) \qquad (24)$$

Dividing Equation (24) by σ_P gives the Percentage Contribution to Total Risk ($\mathrm{PCTR_A}$) for asset i as

$$\mathrm{PCTR_A}(i) = w_P^T(i) \cdot \mathrm{MCTR_A}(i) \,/\, \sigma_P \qquad (25)$$

Similarly, for Active Risk, we can define asset i's Contribution to Active Risk ($\mathrm{CAR_A}$) and Percentage Contribution to Active Risk ($\mathrm{PCAR_A}$) as:

$$\mathrm{CAR_A}(i) = w_{PA}^T(i) \cdot \mathrm{MCAR_A}(i) \qquad (26)$$

$$\mathrm{PCAR_A}(i) = w_{PA}^T(i) \cdot \mathrm{MCAR}(i) \,/\, \sigma_{PA} \qquad (27)$$

Note that each component of $\mathrm{PCAR_A}(i)$ can be positive or negative, such that the sum of contributions from subsets of holdings can exceed one, although the sum over all holdings equals one.

Factor-Level Risk Decomposition

Alternatively, we can decompose the total risk into factor risk and specific risk, using Equation (3), and obtain a decomposition of total risk at the factor level. The total risk can be decomposed as

$$\sigma_P = \frac{x_P^T \cdot F \cdot x_P}{\sigma_P} + \frac{w_P^T \cdot \Delta \cdot w_P}{\sigma_P} \quad (28)$$

The first term, $x_P^T \cdot F \cdot x_P/\sigma_P$, contains the risk attributable to the factors, whereas the second term, $w_P^T \cdot \Delta \cdot w_P/\sigma_P$, contains the risk attributable to the portion of stock-returns that are linearly independent of the factors, and can be attributed to specific sources.

Noting that $x_P^T \cdot F \cdot x_P/\sigma_P = x_P^T \cdot MCTR_F$, the amount of total risk that we can attribute to Factor j, defined as its Contribution to Total Risk (CTR_F), is

$$CTR_F(j) = x_P^T(j) \cdot MCTR_F(j) \quad (29)$$

where $x_P(j)$ is the portfolio's exposure to factor j, as given in Equation (8).

We can divide equation (29) by σ_P, to obtain factor j's Percentage Contribution to Total Risk ($PCTR_F$):

$$PCTR_F(j) = x_P^T(j) \cdot MCTR_F(j) \ / \ \sigma_P \quad (30)$$

Similarly, we define factor j's Contribution to Active Risk (CAR_F) and Percentage Contribution to Active Risk ($PCAR_F$) as:

$$CAR_F(j) = x_{PA}^T(j) \cdot MCAR_F(j) \quad (31)$$

$$PCAR_F(j) = x_{PA}^T(j) \cdot MCAR_F(j) \ / \ \sigma_{PA} \quad (32)$$

Factor-mimicking-portfolio Risk Contribution

While the $MCTR_F$ provides the marginal contribution to risk from a unit change in factor exposure, it does so with the asset holdings assumed held constant. This is fine for decomposing the risk into factor and specific components. However, if one wants to instigate a controlled change in a portfolio's exposure to a particular risk factor, a change in asset holdings is required, which has an additional impact on the risk. We next explore the marginal contribution to risk due to asset-holding changes that are proportional to a factor mimicking portfolio, which we refer to as the Factor-mimicking-portfolio Marginal Contribution to Risk.

Factor-mimicking-portfolio Marginal Risk Contribution

We would like to calculate sensitivities due to changing a portfolio's exposure to particular factors. At the asset level, the marginal contributions capture the change in risk due to changing the holding of just one asset, while leaving all other assets unchanged. By considering factor mimicking portfolios, which produce net exposures that mimic a particular risk factor, we can calculate sensitivities to particular factors.

As discussed previously, the returns from the factor mimicking portfolios mimic the returns of one factor, while effectively having unit net exposure to the particular factor and zero net exposure to every other factor.

To estimate the factor-mimicking-portfolio marginal contribution to total risk, $MCTR_{FMP}$, we add a small portion δ of the factor-k factor mimicking portfolio to a portfolio w_P:

$$w_P \rightarrow w_P + M^T \cdot \delta_k \qquad (33)$$

where the k-th row of M is the 1 x N vector of holdings for the factor-k factor mimicking portfolio, and δ_k is a K x 1 vector containing zeros except in the k-th row, where it contains δ.

To find the effect on risk of adding the factor mimicking portfolio, we multiply the changes in each asset holding in the factor mimicking portfolio by the marginal risk contributions at the asset level, giving the change in portfolio risk as

$$\begin{aligned}\Delta\sigma_P &= [M^T \cdot \delta_k]^T \cdot MCTR_A \qquad (34)\\ &= \delta_k^T \cdot (I - K)(X^T W X)^{-1} X^T W \cdot \left(\frac{V \cdot w_P}{\sigma_P}\right)\end{aligned}$$

where

$$K = (X^T W X)^{-1} C^T (C (X^T W X)^{-1} C^T)^{-1} C$$

so that

$$M = (I - K)(X^T W X)^{-1} X^T W$$

Then the formula for $MCTR_{FMP}$ can be expressed as

$$\begin{aligned}MCTR_{FMP} = \frac{\Delta\sigma_P}{\delta_k^T} &= (I - K)(X^T W X)^{-1} X^T W \cdot \left(\frac{[XFX^T + \Delta] \cdot w_P}{\sigma_P}\right)\\ &= \frac{(I - K)F \cdot x_P}{\sigma_P} + \frac{(I - K)(X^T W X)^{-1} X^T W \Delta \cdot w_P}{\sigma_P}\\ &= (I - K)\left(\frac{F \cdot x_P}{\sigma_P} + \frac{H \cdot w_P}{\sigma_P}\right) \qquad (35)\end{aligned}$$

where

$$H = (X^T W X)^{-1} X^T W \Delta$$

$MCTR_{FMP}(k)$ gives the marginal contribution to total risk due to incrementing the portfolio holdings by the factor-k factor mimicking portfolio.

The F and H terms in Equation (30) respectively are the exposure and holdings weightings that would arise had the risk model regression been solved without constraints. K is the multiplicative and negative incremental effect of including sector and region constraints in the regression. The F term captures the rate of change in factor risk due to additional exposure to the factor mimicking portfolio. The H term captures the change in specific risk due to changing the factor exposures via the factor mimicking portfolio in particular. Empirically, with the exception of highly concentrated portfolios, the second term is much smaller than the first term for the vast majority of funds. Also empirically, the factor K has little effect, so that $MCTR_F$ is a good approximation to $MCTR_{FMP}$.

In a similar way, we can define the factor-mimicking-portfolio marginal contribution to active risk ($MCAR_F$) as:

$$MCAR_{FMP} = \frac{\Delta\sigma_{PA}}{\delta_k^T} = (I - K)\left(\frac{F \cdot x_{PA}}{\sigma_{PA}} + \frac{H \cdot w_{PA}}{\sigma_{PA}}\right) \qquad (36)$$

Estimation of Covariance Matrices

The risk decomposition is performed ex post, looking backwards on realized returns. As such, the aim is to explain where the realized volatility came from. The covariance matrices V, F and Δ are thus estimated from the returns over the span of interest. A minimum span of 3 months is imposed, to avoid highly volatile estimates.

The estimates for V, F and Δ, over the span including days T_1 and T_2, are given by

$$\widehat{\mathrm{V}}(\mathrm{T}_1, \mathrm{T}_2) = \frac{1}{\mathrm{T}_2 - \mathrm{T}_1} \sum_{t=\mathrm{T}_1}^{\mathrm{T}_2} \widetilde{\mathrm{R}}_t \cdot \widetilde{\mathrm{R}}_t^{\mathrm{T}}$$

$$\widehat{\mathrm{F}}(\mathrm{T}_1, \mathrm{T}_2) = \frac{1}{\mathrm{T}_2 - \mathrm{T}_1} \sum_{t=\mathrm{T}_1}^{\mathrm{T}_2} \widetilde{\mathrm{P}}_{\mathrm{F},t} \cdot \widetilde{\mathrm{P}}_{\mathrm{F},t}^{\mathrm{T}}$$

$$\widehat{\Delta}(\mathrm{T}_1, \mathrm{T}_2)_{i,j} = \begin{cases} \dfrac{1}{\mathrm{T}_2 - \mathrm{T}_1} \displaystyle\sum_{t=\mathrm{T}_1}^{\mathrm{T}_2} \widetilde{\mathrm{S}}_{t,i}^2, & i = j \\ 0, & i \neq j \end{cases}$$

where $\widetilde{\mathrm{R}}_t$, $\widetilde{\mathrm{P}}_{\mathrm{F},t}$ and $\widetilde{\mathrm{S}}_t$ have been demeaned over the span, for example,

$$\widetilde{\mathrm{R}}_t = \mathrm{R}_t - \overline{\mathrm{R}} = \mathrm{R}_t - \frac{1}{\mathrm{T}_2 - \mathrm{T}_1} \sum_{t=\mathrm{T}_1}^{\mathrm{T}_2} \mathrm{R}_t$$

and where $\widetilde{\mathrm{R}}_t$, $\widetilde{\mathrm{P}}_{\mathrm{F},t}$ and $\widetilde{\mathrm{S}}_t$ are obtained by fitting the model in Equations (1) and (2) at each time t over the span.

Reference

[1] R. C. Grinold, and R. N. Kahn, "Active portfolio management," 2000.

About Morningstar® Quantitative Research
Morningstar Quantitative Research is dedicated to developing innovative statistical models and data points, including the Morningstar Quantitative Rating, the Quantitative Equity Ratings and the Global Risk Model.

For More Information
+1 312 244-7541
lee.davidson@morningstar.com

MORNINGSTAR®

22 West Washington Street
Chicago, IL 60602 USA