“The Ibbotson-Sinquefield Simulation Made Easy” Made Easy
The curious title of this note refers to Lewis et al. [1980]. In that paper, the authors use a lognormal model of asset returns to proxy the Ibbotson-Sinquefield [1976] simulations, making the Ibbotson-Sinquefield forecasts easy to do. Unfortunately, their paper is not easy to read. This is a condensation of its basic ideas.

In its simplest form, the Ibbotson-Sinquefield forecasting model is that the wealth index on a portfolio follows a lognormal geometric random walk. The wealth index and total rate of return are related by:

\[ W_t = (1 + R_t)W_{t-1} \]

where \( W_t \) is the wealth index at time \( t \) and \( R_t \) is the total rate of return. The \( W \) process is a geometric random if \((1+R_t)\) is independent and identically distributed over all times \( t \). By further assuming that \((1+R_t)\) has a lognormal distribution, we have our model.

Let \( E \) and \( S \) be the expected value and standard deviation of \( R_t \) respectively. Also, let

\[ r_t = \ln(1 + R_t) \]

Let \( m \) and \( s \) be the expected value and standard deviation of \( r_t \) respectively. From the properties of lognormal distributions we have that \( r_t \) is normally distributed and that

\[ \sigma^2 = \ln \left( 1 + \left( \frac{S}{1+E} \right)^2 \right) \]

and

\[ \mu = \ln(1 + E) - \frac{\sigma^2}{2} \]
Consider the value of the wealth index at time $T$. If $W_0$ is one,

$$\ln(W_T) = r_1 + r_2 + \ldots + r_T$$

Thus $\ln(W_T)$ is normally distributed with expected value $m_T$ and standard deviation $\sigma \sqrt{T}$. The $p$’th percentile of $W_T$ is

$$\exp\left(\mu T + z(p) \sigma \sqrt{T}\right)$$

where $z(p)$ is defined by

$$\text{Pr}(Z < z(p)) = p$$

$Z$ being a standard normal random variable. In particular, the 50th percentile (the median) is

$$\text{med}[W_T] = \exp(\mu T)$$

The expected value of $W_T$ is independent of the distribution of $R_T$. So long as the $R_T$’s are independent and are identical in expected value, the expected value of $W_T$ is

$$E[W_T] = (1 + E)^T$$

The geometric mean over periods 1 through $T$, $G_T$, is the rate of the return that compounds up to $W_T$. Thus

$$G_T = \left(1 + W_T\right)^{\frac{1}{T}} - 1$$

So that

$$\ln(1 + G_T) = \frac{\ln(1 + W_T)}{T}$$

Note that $\ln(1+G_T)$ is normally distributed with expected value $m$ and standard deviation $\frac{\sigma}{\sqrt{T}}$. 

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The $p$’th percentile of $G_T$ is

$$\exp\left(\mu + z(p)\frac{\sigma}{\sqrt{T}}\right) - 1$$

In particular, the median is

$$\text{med}[G_T] = \exp(\mu) - 1 = \frac{(1 + E)^2}{\sqrt{(1 + E)^2 + S^2}} - 1$$

The expected value of $G_T$ is

$$E[G_T] = \exp\left(\mu + \frac{1}{2} \frac{\sigma^2}{T}\right) - 1 = (1 + \text{med}[G_T])\sqrt{\frac{1 + E}{1 + \text{med}[G_T]}} - 1$$

The median of $W_T$ can be expressed in terms of the median of $G_T$; namely,

$$\text{med}[W_T] = (1 + \text{med}[G_T])^T$$

The parameters $E$ and $S$ can be estimated from historical data. In particular, $E$ should be estimated from historical arithmetic means, not geometric means. The geometric mean is a measure of performance, past or future. When forecasting its future value, $G_T$, we use arithmetic means to form inputs for forecasting.
References
